Deformations of singularities 2×2 traceless linear differential systems

Martin Klimeš

Bifurcations of Dynamical Systems and Numerics, Zagreb, May 9–11, 2023

Supported by FRABDYN, Croatian Science Foundation project PZS3055

Meromorphic connection over a Riemann surface

Locally near a pole:

$$\nabla(x) = d - A(x)\frac{dx}{x^{k+1}}, \qquad A(0) \neq 0, \quad k = \text{Poincaré rank.}$$

Linear differential system: $x^{k+1}\frac{d}{dx}y = A(x)y$.

Transformations: formal / analytic

- *gauge transformations:*

$$y \mapsto T(x)y, \qquad \det T(0) \neq 0,$$

- *gauge–coordinate transformations:*

$$x \mapsto \phi(x), \quad y \mapsto T(x)y, \qquad \tfrac{d}{dx}\phi(0) \neq 0, \quad \det T(0) \neq 0,$$

Meromorphic connection over a Riemann surface

Locally near a pole:

$$\nabla(x) = d - A(x)\frac{dx}{x^{k+1}}, \qquad A(0) \neq 0, \quad k = \text{Poincaré rank.}$$

Linear differential system: $x^{k+1}\frac{d}{dx}y = A(x)y$.

Transformations: formal / analytic

- *gauge transformations:*

$$y \mapsto T(x)y, \qquad \det T(0) \neq 0,$$

- *gauge–coordinate transformations:*

$$x \mapsto \phi(x), \quad y \mapsto T(x)y, \qquad \tfrac{d}{dx}\phi(0) \neq 0, \quad \det T(0) \neq 0,$$

Meromorphic connection over a Riemann surface

Locally near a pole:

$$\nabla(x) = d - A(x)\frac{dx}{x^{k+1}}, \qquad A(0) \neq 0, \quad k = \text{Poincaré rank.}$$

Linear differential system: $x^{k+1}\frac{d}{dx}y = A(x)y$.

Transformations: formal / analytic

- *gauge transformations:*

$$y \mapsto T(x)y, \qquad \det T(0) \neq 0,$$

- *gauge–coordinate transformations:*

$$x \mapsto \phi(x), \quad y \mapsto T(x)y, \qquad \tfrac{d}{dx}\phi(0) \neq 0, \quad \det T(0) \neq 0,$$

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

1. Formal invariants:

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$

  = *meromorphic 1-forms:*

  $$\lambda_i(x)\frac{dx}{x^{k+1}} = (\lambda_{i,0} + \ldots + \lambda_{i,k}x^k)\frac{dx}{x^{k+1}},$$

- [Hukuhara, Turittin, Levelt]: over $\mathbb{C}((x^{1/p}))$ for some $p \geq 1$

- [Balser, Jurkat, Lutz]: canonical form of formal fundamental solution

  $$\hat{F}(x)\,P(x^{-1})\,x^K x^J U\,e^{Q(x^{-1})}.$$

2. Additional analytic invariants if $k > 0$: Stokes matrices

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

1. *Formal invariants:*

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$

  = *meromorphic 1-forms:*

$$\lambda_i(x)\frac{dx}{x^{k+1}} = (\lambda_{i,0} + \ldots + \lambda_{i,k}x^k)\frac{dx}{x^{k+1}},$$

- [Hukuhara, Turittin, Levelt]: over $\mathbb{C}((x^{1/p}))$ for some $p \geq 1$
- [Balser, Jurkat, Lutz]: canonical form of formal fundamental solution

$$\hat{F}(x)\, P(x^{-1})\, x^K x^J U\, e^{Q(x^{-1})}.$$

2. *Additional analytic invariants* if $k > 0$: Stokes matrices

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

1. Formal invariants:

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$

  = *meromorphic 1-forms:*

  $$\lambda_i(x)\frac{dx}{x^{k+1}} = (\lambda_{i,0} + \ldots + \lambda_{i,k}x^k)\frac{dx}{x^{k+1}},$$

- [Hukuhara, Turittin, Levelt]: over $\mathbb{C}((x^{1/p}))$ for some $p \geq 1$

- [Balser, Jurkat, Lutz]: canonical form of formal fundamental solution

  $$\hat{F}(x)\,P(x^{-1})\,x^K x^J U\,e^{Q(x^{-1})}.$$

2. Additional analytic invariants if $k > 0$: Stokes matrices

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

1. Formal invariants:

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$

  = *meromorphic 1-forms:*

  $$\lambda_i(x)\frac{dx}{x^{k+1}} = (\lambda_{i,0} + \ldots + \lambda_{i,k}x^k)\frac{dx}{x^{k+1}},$$

- [Hukuhara, Turittin, Levelt]: over $\mathbb{C}((x^{1/p}))$ for some $p \geq 1$

- [Balser, Jurkat, Lutz]: canonical form of formal fundamental solution

  $$\hat{F}(x)\, P(x^{-1})\, x^K x^J U\, e^{Q(x^{-1})}.$$

2. Additional analytic invariants if $k > 0$: Stokes matrices

Local analytic classification

Birkhoff (1913), Hukuhara (1937), Turittin (1955), Levelt (1961), Sibuya (1967), Malgrange (1971), Balser, Jurkat, Lutz (1979), Ramis (1985), ...

1. Formal invariants:

- non-resonant case: polar parts of the eigenvalues of $A(x)\frac{dx}{x^{k+1}}$

  = *meromorphic 1-forms:*

  $$\lambda_i(x)\frac{dx}{x^{k+1}} = (\lambda_{i,0} + \ldots + \lambda_{i,k}x^k)\frac{dx}{x^{k+1}},$$

- [Hukuhara, Turittin, Levelt]: over $\mathbb{C}((x^{1/p}))$ for some $p \geq 1$

- [Balser, Jurkat, Lutz]: canonical form of formal fundamental solution

  $$\hat{F}(x)\,P(x^{-1})\,x^K x^J U\,e^{Q(x^{-1})}.$$

2. Additional analytic invariants if $k > 0$: Stokes matrices

Confluence

Parametric family

$$\nabla_\epsilon(x) = d - A(x,\epsilon)\frac{dx}{P(x,\epsilon)}, \qquad P(x,0) = x^{k+1}$$

- *Degeneration through confluence of singularities:*

  $$P(x,\epsilon) = \Pi_i (x - a_i(\epsilon))^{k_i+1} \to P(x,0) = x^{k+1}$$

  [Hurtubise, Lambert, Rousseau (2014)] under non-resonance condition

- *Degeneration through confluence of eigenvalues:* $P(x,\epsilon) = x^{k+1}$, order of $\big(\lambda_i(x,\epsilon) - \lambda_j(x,\epsilon)\big)\frac{dx}{x^{k+1}}$ increases at $\epsilon = 0$

Problem: Explain the relation between the Stokes & mondromy data of ∇_ϵ and those of ∇_0.

Confluence

Parametric family

$$\nabla_\epsilon(x) = d - A(x,\epsilon)\frac{dx}{P(x,\epsilon)}, \qquad P(x,0) = x^{k+1}$$

- *Degeneration through confluence of singularities:*

  $$P(x,\epsilon) = \Pi_i (x - a_i(\epsilon))^{k_i+1} \to P(x,0) = x^{k+1}$$

  [Hurtubise, Lambert, Rousseau (2014)] under non-resonance condition

- *Degeneration through confluence of eigenvalues:* $P(x,\epsilon) = x^{k+1}$, order of $\big(\lambda_i(x,\epsilon) - \lambda_j(x,\epsilon)\big)\frac{dx}{x^{k+1}}$ increases at $\epsilon = 0$

Problem: Explain the relation between the Stokes & mondromy data of ∇_ϵ and those of ∇_0.

Confluence

Parametric family

$$\nabla_\epsilon(x) = d - A(x,\epsilon)\frac{dx}{P(x,\epsilon)}, \qquad P(x,0) = x^{k+1}$$

- *Degeneration through confluence of singularities:*

  $$P(x,\epsilon) = \Pi_i(x - a_i(\epsilon))^{k_i+1} \to P(x,0) = x^{k+1}$$

  [Hurtubise, Lambert, Rousseau (2014)] under non-resonance condition

- *Degeneration through confluence of eigenvalues:* $P(x,\epsilon) = x^{k+1}$, order of $\big(\lambda_i(x,\epsilon) - \lambda_j(x,\epsilon)\big)\frac{dx}{x^{k+1}}$ increases at $\epsilon = 0$

Problem: Explain the relation between the Stokes & mondromy data of ∇_ϵ and those of ∇_0.

Confluence

Parametric family

$$\nabla_\epsilon(x) = d - A(x,\epsilon)\frac{dx}{P(x,\epsilon)}, \qquad P(x,0) = x^{k+1}$$

- *Degeneration through confluence of singularities:*

  $$P(x,\epsilon) = \Pi_i(x - a_i(\epsilon))^{k_i+1} \to P(x,0) = x^{k+1}$$

  [Hurtubise, Lambert, Rousseau (2014)] under non-resonance condition

- *Degeneration through confluence of eigenvalues:* $P(x,\epsilon) = x^{k+1}$, order of $\big(\lambda_i(x,\epsilon) - \lambda_j(x,\epsilon)\big)\frac{dx}{x^{k+1}}$ increases at $\epsilon = 0$

Problem: Explain the relation between the Stokes & mondromy data of ∇_ϵ and those of ∇_0.

Traceless meromorphic connection on rank 2 vector bundle

Locally traceless 2×2 system:

$$x^{k+1} \tfrac{d}{dx} y = A(x) y, \qquad A(x) \in \mathfrak{sl}_2(\mathbb{C}),\ A(0) \neq 0.$$

Lemma

The system is analytically gauge equivalent to a companion system

$$x^{k+1} \tfrac{d}{dx} y = \left(\begin{smallmatrix} 0 & 1 \\ Q(x) & 0 \end{smallmatrix}\right) y,$$

associated to the second order LDE

$$\left(x^{k+1} \tfrac{d}{dx}\right)^{\circ 2} y_1 = Q(x)\, y_1.$$

Meromorphic quadratic differential

$$Q(x) \left(\frac{dx}{x^{k+1}}\right)^2 = -\det \left[\left(\begin{smallmatrix} 0 & 1 \\ Q(x) & 0 \end{smallmatrix}\right) \frac{dx}{x^{k+1}} \right]$$

$$m = \operatorname{ord}_{x=0} Q(x), \quad \max\{k - \tfrac{m}{2},\ 0\} = \textit{Katz rank}.$$

Traceless meromorphic connection on rank 2 vector bundle

Locally traceless 2×2 system:

$$x^{k+1} \tfrac{d}{dx} y = A(x) y, \qquad A(x) \in \mathfrak{sl}_2(\mathbb{C}),\ A(0) \neq 0.$$

Lemma

The system is analytically gauge equivalent to a companion system

$$x^{k+1} \tfrac{d}{dx} y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y,$$

associated to the second order LDE

$$\left(x^{k+1} \tfrac{d}{dx}\right)^{\circ 2} y_1 = Q(x)\, y_1.$$

Meromorphic quadratic differential

$$Q(x) \left(\frac{dx}{x^{k+1}}\right)^2 = -\det \left[\begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} \frac{dx}{x^{k+1}} \right]$$

$$m = \mathrm{ord}_{x=0}\, Q(x), \quad \max\{k - \tfrac{m}{2},\ 0\} = \textit{Katz rank}.$$

Traceless meromorphic connection on rank 2 vector bundle

Locally traceless 2×2 system:

$$x^{k+1} \tfrac{d}{dx} y = A(x) y, \qquad A(x) \in \mathfrak{sl}_2(\mathbb{C}),\ A(0) \neq 0.$$

Lemma

The system is analytically gauge equivalent to a companion system

$$x^{k+1} \tfrac{d}{dx} y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y,$$

associated to the second order LDE

$$\left(x^{k+1} \tfrac{d}{dx}\right)^{\circ 2} y_1 = Q(x)\, y_1.$$

Meromorphic quadratic differential

$$Q(x) \left(\frac{dx}{x^{k+1}}\right)^2 = -\det \left[\begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} \frac{dx}{x^{k+1}} \right]$$

$$m = \operatorname{ord}_{x=0} Q(x), \quad \max\{k - \tfrac{m}{2},\ 0\} = \textit{Katz rank}.$$

Formal classification

Theorem (K.)
Two companion systems $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$ *with the same* $k \geq 0$ *are formally gauge equivalent (resp. formally gauge–coordinate equivalent) iff they define the same (resp. equivalent) jet*

$$\left(j^N Q(x)\right)\left(\frac{dx}{x^{k+1}}\right)^2, \qquad N = \begin{cases} k & \text{if } m = 0, \\ k+m-1 & \text{if } 0 < m \leq 2k, \\ 3k & \text{if } 2k < m, \end{cases}$$

$$m = \operatorname{ord}_{x=0} Q(x),$$

& have conjugated monodromies if $m = 2k$*.*

Formal classification

Formal normal form (w.r.t. formal gauge–coordinate transformations):

$$Q_{\mathrm{nf}}\left(\frac{dx}{x^{k+1}}\right)^2 = \begin{cases} x^m\left(1+2\sqrt{\mu}\,x^{k-\frac{m}{2}}\right)\left(\frac{dx}{x^{k+1}}\right)^2, & \\ \mu x^{2k}\left(\frac{dx}{x^{k+1}}\right)^2, & m = 2k, \\ \left(\mu x^{2k}+x^{2k+l}\right)\left(\frac{dx}{x^{k+1}}\right)^2, & m = 2k, \quad \mu = \frac{l^2-k^2}{4}, \\ 0, & m > 3k. \end{cases}$$

where

$$\mu = \mathrm{res}^2_{x=0}\, Q(x)\left(\frac{dx}{x^{k+1}}\right)^2 := \left(\mathrm{res}_{x=0}\sqrt{Q(s)}\frac{dx}{x^{k+1}}\right)^2 \ \ldots \ \textit{square residue},$$

$\mu = 0$ if m is odd or $m > 2k$.

Stokes geometry (irregular case $m < 2k$)

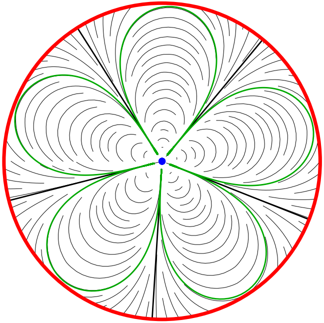
Let $\tilde{Q} = j^N Q, \quad m = \text{ord}_{x=0}\, Q.$

Horizontal foliation of

$$\tilde{Q}(x)\left(\frac{dx}{x^{k+1}}\right)^2,$$

= real time trajectories of

$$\frac{dx}{dt} = \pm \quad \frac{x^{k+1}}{\sqrt{\tilde{Q}(x)}}, \qquad t \in \mathbb{R}.$$



Complete real trajectories inside a disc $\mathbb{D}$:
organized in $2k - m$ *petals*.

(anti)-Stokes regions: trajectories that leave $\mathbb{D}$.

Stokes geometry (irregular case $m < 2k$)

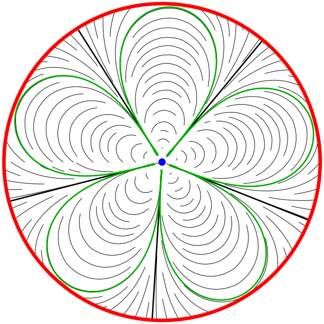
Let $\tilde{Q} = j^N Q, \quad m = \mathrm{ord}_{x=0}\, Q.$

Horizontal foliation of

$$e^{-2i\theta}\, \tilde{Q}(x)\left(\frac{dx}{x^{k+1}}\right)^2, \quad |\theta| < \frac{\pi}{2},$$

= real time trajectories of

$$\frac{dx}{dt} = \pm e^{i\theta}\, \frac{x^{k+1}}{\sqrt{\tilde{Q}(x)}}, \qquad t \in \mathbb{R}.$$



Complete real trajectories inside a disc $\mathbb{D}$:
organized in $2k - m$ *petals*.

(anti)-Stokes regions: trajectories that leave $\mathbb{D}$.

Stokes geometry (irregular case $m < 2k$)

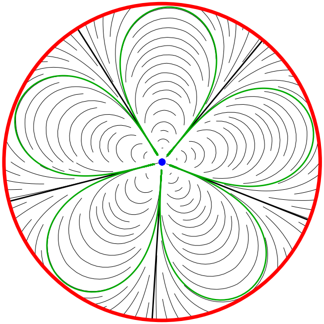
Let $\tilde{Q} = j^N Q, \quad m = \text{ord}_{x=0} Q.$

Horizontal foliation of

$$e^{-2i\theta}\,\tilde{Q}(x)\left(\frac{dx}{x^{k+1}}\right)^2, \quad |\theta| < \frac{\pi}{2},$$

= real time trajectories of

$$\frac{dx}{dt} = \pm e^{i\theta}\,\frac{x^{k+1}}{\sqrt{\tilde{Q}(x)}}, \qquad t \in \mathbb{R}.$$

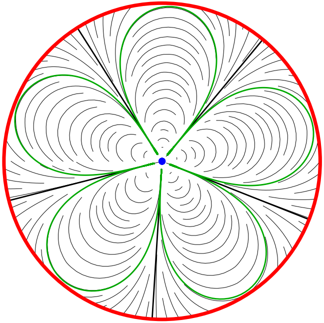


Complete real trajectories inside a disc $\mathbb{D}$:
organized in $2k - m$ *petals*.

(anti)-Stokes regions: trajectories that leave $\mathbb{D}$.

Stokes geometry (irregular case $m < 2k$)

Let $\tilde{Q} = j^N Q, \quad m = \mathrm{ord}_{x=0} Q.$



Horizontal foliation of

$$e^{-2i\theta}\, \tilde{Q}(x) \Big(\tfrac{dx}{x^{k+1}} \Big)^2, \quad |\theta| < \tfrac{\pi}{2},$$

= real time trajectories of

$$\frac{dx}{dt} = \pm e^{i\theta} \frac{x^{k+1}}{\sqrt{\tilde{Q}(x)}}, \qquad t \in \mathbb{R}.$$

Complete real trajectories inside a disc $\mathbb{D}$:
organized in $2k - m$ *petals*.

(anti)-Stokes regions: trajectories that leave $\mathbb{D}$.

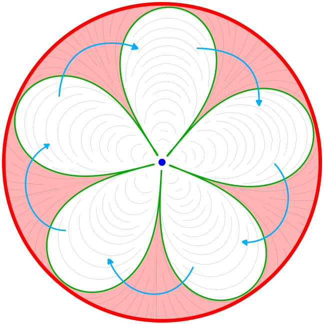
Theorem (Birkhoff, Hukuhara, Turittin, Sibuya)
System $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$:

1. *Half-trajectory* ⇝ *1-dimensional space of subdominant solutions (those with vanishing limit along the trajectory),*

2. *Complete trajectory* ⇝ *direct decomposition of solution space*

   ⇝ *basis of solutions on petals*

Stokes matrices: change of basis when crossing (anti)-Stokes regions



Theorem (Birkhoff, Hukuhara, Turittin, Sibuya)

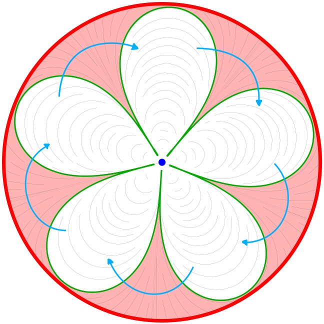
System $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$*:*

1. *Half-trajectory* $\rightsquigarrow$ *1-dimensional space of subdominant solutions (those with vanishing limit along the trajectory),*

2. *Complete trajectory* $\rightsquigarrow$ *direct decomposition of solution space*

   $\rightsquigarrow$ *basis of solutions on petals*

Stokes matrices: change of basis when crossing (anti)-Stokes regions



Theorem (Birkhoff, Hukuhara, Turittin, Sibuya)

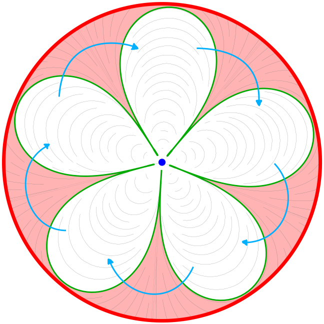
System $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$*:*

1. *Half-trajectory* $\rightsquigarrow$ *1-dimensional space of subdominant solutions (those with vanishing limit along the trajectory),*

2. *Complete trajectory* $\rightsquigarrow$ *direct decomposition of solution space*

   $\rightsquigarrow$ *basis of solutions on petals*

Stokes matrices: change of basis when crossing (anti)-Stokes regions



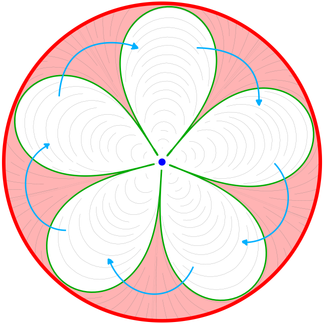
Theorem (Birkhoff, Hukuhara, Turittin, Sibuya)
System $x^{k+1}\frac{d}{dx}y = \begin{pmatrix} 0 & 1 \\ Q(x) & 0 \end{pmatrix} y$*:*

1. *Half-trajectory* $\rightsquigarrow$ *1-dimensional space of subdominant solutions (those with vanishing limit along the trajectory),*

2. *Complete trajectory* $\rightsquigarrow$ *direct decomposition of solution space*

   $\rightsquigarrow$ *basis of solutions on petals*

Stokes matrices: change of basis when crossing (anti)-Stokes regions



Parametric families: confluence

Up to analytic gauge transformation (analytic in parameter):

$$P(x,\epsilon)\frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ Q(x,\epsilon) & 0 \end{pmatrix} y, \qquad P(x,0) = x^{k+1}, \qquad \epsilon \in (\mathbb{C}^N, 0).$$

Theorem (K.)

Formal gauge invariants, w.r.t. $\mathbb{C}[[x,\epsilon]]$, when $m = \operatorname{ord}_{x=0} Q(x,0) \in \{0,1\}$:

$$\Big(Q(x,\epsilon) \mod P(x,\epsilon)\Big)\left(\frac{dx}{P(x,\epsilon)}\right)^2$$

Formal gauge–coordinate invariants

$$Q_{nf}(x,\epsilon)\left(\frac{dx}{P_{nf}(x,\epsilon)^2}\right)^2 = \begin{cases} \left(1+2\sqrt{\mu(\epsilon)}x^k\right)\left(\frac{dx}{x^{k+1}+p_{k-1}(\epsilon)x^{k-1}+\ldots+p_0(\epsilon)}\right)^2, & m=0, \\ \left(q_0(\epsilon)+x\right)\left(\frac{dx}{x^{k+1}+p_{k-1}(\epsilon)x^{k-1}+\ldots+p_0(\epsilon)}\right)^2, & m=1, \end{cases}$$

unique up to rotations $x \mapsto e^{\frac{2j\pi i}{2k-m}}x$, $j \in \mathbb{Z}_{2k-m}$.

Parametric families: confluence

Up to analytic gauge transformation (analytic in parameter):

$$P(x,\epsilon)\frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ Q(x,\epsilon) & 0 \end{pmatrix} y, \qquad P(x,0) = x^{k+1}, \qquad \epsilon \in (\mathbb{C}^N, 0).$$

Theorem (K.)

Formal gauge invariants, w.r.t. $\mathbb{C}[[x,\epsilon]]$*, when* $m = \operatorname{ord}_{x=0} Q(x,0) \in \{0,1\}$*:*

$$\Big(Q(x,\epsilon) \mod P(x,\epsilon)\Big)\left(\frac{dx}{P(x,\epsilon)}\right)^2$$

Formal gauge–coordinate invariants

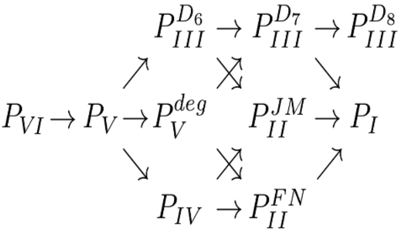
$$Q_{nf}(x,\epsilon)\left(\frac{dx}{P_{nf}(x,\epsilon)^2}\right)^2 = \begin{cases} \left(1+2\sqrt{\mu(\epsilon)}x^k\right)\left(\frac{dx}{x^{k+1}+p_{k-1}(\epsilon)x^{k-1}+\ldots+p_0(\epsilon)}\right)^2, & m=0, \\ \left(q_0(\epsilon)+x\right)\left(\frac{dx}{x^{k+1}+p_{k-1}(\epsilon)x^{k-1}+\ldots+p_0(\epsilon)}\right)^2, & m=1, \end{cases}$$

unique up to rotations $x \mapsto e^{\frac{2j\pi i}{2k-m}}x$*,* $j \in \mathbb{Z}_{2k-m}$*.*

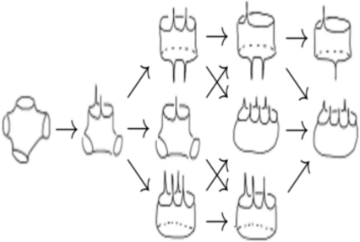
Example: degeneration of Painlevé equations

Confluences of isomonodromic deformation systems

- isomonodromic parameter t,
- confluence parameter ϵ.



(a) Painlevé equations



(b) Isomonodromic deformations

[Ohyama, Okumura (2006), Chekhov, Mazzocco, Roubtsov (2017)]

Horizontal foliation

Meromorphic quadratic differential $\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \quad \tilde{Q} = Q \mod P.$

Saddles = zeros of order ≥ -1 of $\frac{\tilde{Q}(x,\epsilon)}{P(x,\epsilon)^2}$.

Horizontal foliation of

$$e^{-2i\theta}\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation } |\theta| < \tfrac{\pi}{2},$$

Rotationally stable inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.

Complete trajectories inside $\mathbb{D}$ organized in *zones*:

- *sepal zones*: attached to a single parabolic equilibrium.
- *$\alpha\omega$-zones*: attached to a pair of equilibria.
  $\rightsquigarrow$ split into two parts by a *gate*.

$$\textit{Zones:} \begin{cases} \textit{outer}: & \text{touch } \partial\mathbb{D}, \\ \textit{inner}: & \text{touch a neighborhood of a saddle inside } \mathbb{D} \end{cases}$$

Horizontal foliation

Meromorphic quadratic differential $\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \quad \tilde{Q} = Q \mod P.$

Saddles = zeros of order ≥ -1 of $\frac{\tilde{Q}(x,\epsilon)}{P(x,\epsilon)^2}$.

Horizontal foliation of

$$e^{-2i\theta}\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation } |\theta| < \tfrac{\pi}{2},$$

Rotationally stable inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.

Complete trajectories inside $\mathbb{D}$ organized in *zones*:

- *sepal zones*: attached to a single parabolic equilibrium.
- *$\alpha\omega$-zones*: attached to a pair of equilibria.
  $\rightsquigarrow$ split into two parts by a *gate*.

Zones: $\begin{cases} \textit{outer}: & \text{touch } \partial\mathbb{D}, \\ \textit{inner}: & \text{touch a neighborhood of a saddle inside } \mathbb{D} \end{cases}$

Horizontal foliation

Meromorphic quadratic differential $\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \quad \tilde{Q} = Q \mod P.$

Saddles = zeros of order ≥ -1 of $\frac{\tilde{Q}(x,\epsilon)}{P(x,\epsilon)^2}$.

Horizontal foliation of

$$e^{-2i\theta}\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation } |\theta| < \frac{\pi}{2},$$

Rotationally stable inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.

Complete trajectories inside $\mathbb{D}$ organized in *zones*:

- *sepal zones*: attached to a single parabolic equilibrium.
- *$\alpha\omega$-zones*: attached to a pair of equilibria.
  $\rightsquigarrow$ split into two parts by a *gate*.

Zones: $\begin{cases} \textit{outer}: & \text{touch } \partial\mathbb{D}, \\ \textit{inner}: & \text{touch a neighborhood of a saddle inside } \mathbb{D} \end{cases}$

Horizontal foliation

Meromorphic quadratic differential $\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \quad \tilde{Q} = Q \mod P.$

Saddles = zeros of order ≥ -1 of $\frac{\tilde{Q}(x,\epsilon)}{P(x,\epsilon)^2}$.

Horizontal foliation of

$$e^{-2i\theta}\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation } |\theta| < \tfrac{\pi}{2},$$

Rotationally stable inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.

Complete trajectories inside $\mathbb{D}$ organized in *zones*:

- *sepal zones*: attached to a single parabolic equilibrium.
- *$\alpha\omega$-zones*: attached to a pair of equilibria.
  $\rightsquigarrow$ split into two parts by a *gate*.

Zones: $\begin{cases} \textit{outer}: & \text{touch } \partial\mathbb{D}, \\ \textit{inner}: & \text{touch a neighborhood of a saddle inside } \mathbb{D} \end{cases}$

Horizontal foliation

Meromorphic quadratic differential $\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \quad \tilde{Q} = Q \mod P.$

Saddles = zeros of order ≥ -1 of $\frac{\tilde{Q}(x,\epsilon)}{P(x,\epsilon)^2}$.

Horizontal foliation of

$$e^{-2i\theta}\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2, \qquad \text{angle of rotation } |\theta| < \tfrac{\pi}{2},$$

Rotationally stable inside $\mathbb{D}$ if no trajectory leaves $\mathbb{D} \setminus$ Saddles in both directions.

Complete trajectories inside $\mathbb{D}$ organized in *zones*:

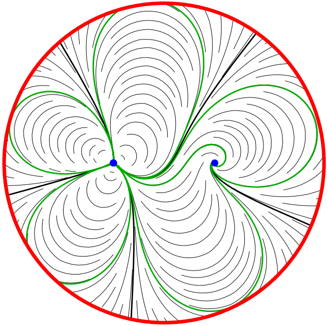
- *sepal zones*: attached to a single parabolic equilibrium.
- *$\alpha\omega$-zones*: attached to a pair of equilibria.
  $\rightsquigarrow$ split into two parts by a *gate*.

Zones: $\begin{cases} \textit{outer}: & \text{touch } \partial\mathbb{D}, \\ \textit{inner}: & \text{touch a neighborhood of a saddle inside } \mathbb{D} \end{cases}$

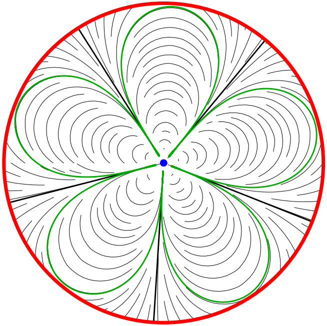
Example

$$e^{-2i\theta}\,\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2 = e^{-2i\theta}\,x\left(\frac{dx}{x^3(x-\epsilon)}\right)^2$$

- 5 outer zones (3 sepal zones + 2 parts of an $\alpha\omega$-zone)



(a) $\epsilon \neq 0$

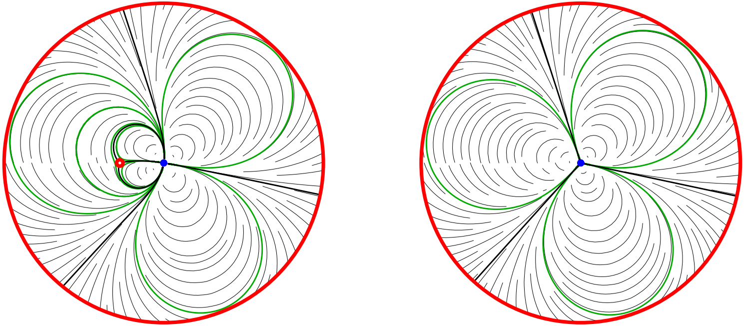


(b) $\epsilon = 0$

Example

$$e^{-2i\theta}\,\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2 = e^{-2i\theta}\,x\left(\frac{dx}{(x-\epsilon)^3}\right)^2$$

- 3 outer zones + 3 inner zones (4 sepal zones + 2 parts of an $\alpha\omega$-zone)

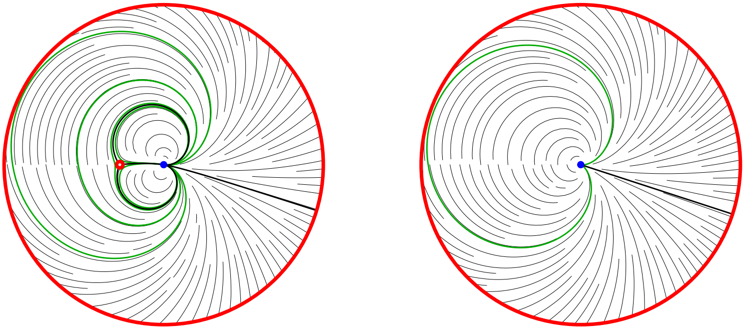


(a) $\epsilon \neq 0$ (b) $\epsilon = 0$

Example

$$e^{-2i\theta}\,\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2 = e^{-2i\theta}\,x\left(\frac{dx}{(x-\epsilon)^3}\right)^2$$

- 1 outer zone + 3 inner zones (2 sepal zones + 2 parts of an $\alpha\omega$-zone)

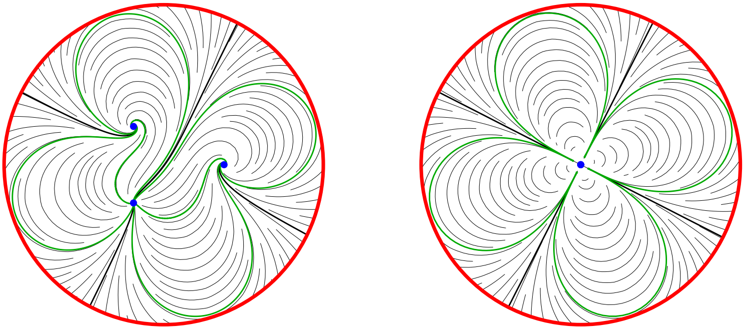


(a) $\epsilon \neq 0$ (b) $\epsilon = 0$

Example

$$e^{-2i\theta}\,\tilde{Q}(x,\epsilon)\left(\frac{dx}{P(x,\epsilon)}\right)^2 = e^{-2i\theta}\left(\frac{dx}{x^3+\epsilon_1 x+\epsilon_0}\right)^2$$

- 4 outer zones (4 parts of 2 $\alpha\omega$-zones)



(a) $\epsilon \neq 0$ (b) $\epsilon = 0$

Confluent Stokes geometry

Assume $m \in \{0, 1\}$.

Theorem (Levinson (1948))

Landing half-trajectories $\rightsquigarrow$ 1-dimensional spaces of subdominant solutions.

Theorem

If either

- *(no saddle): $\frac{P(x,\epsilon)}{Q(x,\epsilon)}$ is analytic,*
- *(only 1 singularity): $P(x,\epsilon) = x^{k+1}$,*

then for rotationally stable (ϵ, θ):

complete trajectories $\rightsquigarrow$ mixed solution bases:

$$\begin{cases} \text{outer:} & \text{normalized have a limit when } \epsilon \to 0 \text{ radially,} \\ \text{inner:} & \text{disappear at the limit.} \end{cases}$$

Confluent Stokes geometry

Assume $m \in \{0, 1\}$.

Theorem (Levinson (1948))

Landing half-trajectories $\rightsquigarrow$ 1-dimensional spaces of subdominant solutions.

Theorem

If either

- *(no saddle):* $\frac{P(x,\epsilon)}{Q(x,\epsilon)}$ *is analytic,*
- *(only 1 singularity):* $P(x,\epsilon) = x^{k+1}$,

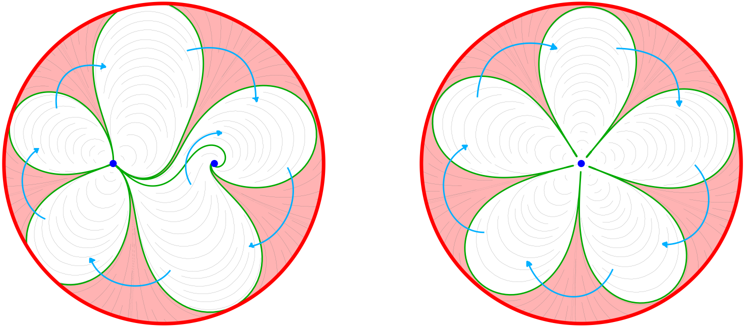
then for rotationally stable (ϵ, θ)*:*

complete trajectories $\rightsquigarrow$ mixed solution bases:

$$\begin{cases} \text{outer:} & \text{normalized have a limit when } \epsilon \to 0 \text{ radially,} \\ \text{inner:} & \text{disappear at the limit.} \end{cases}$$

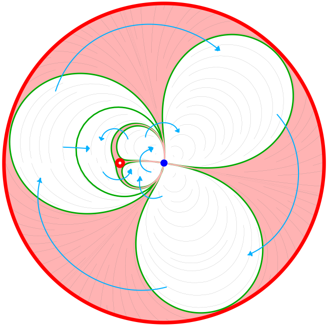
Confluent Stokes geometry

- *Stokes matrices:* change of solution bases when crossing (anti)-Stokes regions.
- *Formal monodromies:* change of solution bases when crossing gates of $\alpha\omega$-zones.

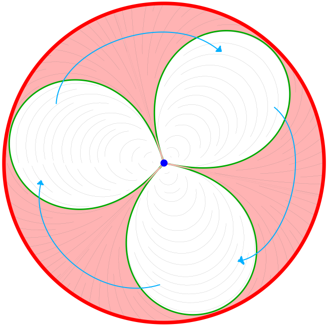


(a) $\epsilon \neq 0$ (b) $\epsilon = 0$

Confluent Stokes geometry



(a) $\epsilon \neq 0$



(b) $\epsilon = 0$